JOURNAL OF APPROXIMATION THEORY **40**, 384–390 (1984)

On Incomplete Polynomials*

BOGDAN M. BAISHANSKI

Department of Mathematics, Ohio State University, Columbus, Ohio 43210, U.S.A.

Communicated by Oved Shisha

Received April 14, 1983

Approximation of a monomial by incomplete polynomials was the subject of several investigations. Here the setting is inverted (approximating a polynomial by monomials) and generalized (approximating a polynomial by incomplete polynomials).

INTRODUCTION

For any l-tuple $\bar{n} = (n_1, n_2,..., n_l)$ of integers, where $0 \leqslant n_1 < n_2 < \cdots < n_l$, the vector-space spanned by $x^{n_1}, x^{n_2},..., n^{n_l}$ is denoted $V_{\bar{n}}$. Let S denote a set of nonnegative integers; the set of l-tuples $\bar{n}$ such that $n_j \in S$ for $j = 1, 2,..., l$, and $n_j < n_{j+1}$ for $j = 1,..., l-1$, is denoted S_l. It is always assumed that $l \leqslant \operatorname{card}(S)$. We denote $\bigcup \{V_{\bar{n}} \mid \bar{n} \in S_l\}$ by $\pi_l(S)$; the set of all polynomials of degree $\leqslant d$ is denoted $\mathscr{P}_d$; the length of a polynomial P is the number of its nonzero coefficients.

The norm $\| \ \|$ is always the L_p-norm on $[a, b]$, where $1 \leqslant p \leqslant \infty$, $0 \leqslant a < b < \infty$; however, if $p = \infty$, it is assumed that $a > 0$. The function f belongs to $L_p(a, b)$ if $p < \infty$, and to $C[a, b]$ if $p = \infty$. It is known [1] that

$$d_{\bar{n}}(f) = \min\{\|f - y\| \mid y \in V_{\bar{n}}\}$$

attains its minimum for some $\bar{n} = \bar{n}(f) \in S_l$ (that $\bar{n}(f)$ is generally not unique). In other words, there exists $\bar{n}(f) \in S_l$ such that

$$d_{\bar{n}(f)}(f) = \inf\{\|f - y\| \mid y \in \pi_l(S)\}.$$

Thanks to the work of several mathematicians [5, 3, 9, 8] we have pretty good information about $n(f)$ in case $f(x) = x^k$; for example, the following result is known:

$$\begin{aligned} &\text{if } S = \{n \mid n = 0, 1, 2,..., n \neq k\} \text{ and } \bar{n}(x^k) = (n_1, n_2,..., n_l), \\ &\text{then the set } \{n_1, n_2,..., n_l\} \cup \{k\} \text{ is a set of consecutive integers.} \end{aligned} \tag{1}$$

* A preliminary version of this paper was communicated at the Fourth International Symposium on Approximation Theory, held at A & M University, College Station, Texas, January 1983.

0021-9045/84 $3.00

In case $f(x) = x^k$, information is available also about $d_{\bar{n}(f)}(f)$, [4, 6, 7, 2]; but that is a topic with which we shall not be concerned here.

In this paper we study $\bar{n}(P)$, where

$$P = P(x) = \sum_{j=1}^{k} c_j x^{p_j}, \qquad 0 \leqslant p_1 < \cdots < p_k. \tag{2}$$

2. Initial Remarks

(i) The following example shows that if x^k is replaced by a polynomial P, a straightforward generalization of (1) will not hold, not even in some of the simplest cases. Let

$$P(x) = x^k - \gamma x^{k-1}, \qquad \gamma = k/(k+1),$$

and $d_n(P) = \min_c \|P(x) - cx^n\|_{L_2(0,1)}$. In view of (1), one would expect that the sequence $\{d_n\}_{n=0}^{+\infty}$ takes its minimum value at one of the integers $n = k-2,\ k-1,\ k,\ k+1$. However,

$$d_n^2(f) = (P, P) - \frac{1}{(x^n, x^n)}(x^n, P)^2 = \|P\|^2 - (\sqrt{2n+1}\,(x^n, P))^2, \tag{3}$$

so that $d_n(P)$ will be minimum when

$$\sqrt{2n+1}\,|(x^n, P)| = \sqrt{2n+1}\left|\frac{1}{n+k+1} - \frac{k}{k+1}\frac{1}{n+k}\right|$$

is maximum. The last expression is equal to

$$\frac{1}{k+1}\varphi_k(n), \qquad \text{where} \quad \varphi_k(t) = \frac{t\sqrt{2t+1}}{(t+k+1)(t+k)}.$$

It is easy to check that $\varphi_k'(t)$ has only one zero for $t > 0$, that $\varphi_k'(3k) > 0$, $\varphi_k'(3k+1) < 0$, and that $\varphi_k(3k+1) > \varphi_k(3k)$. It follows that $d_n(P)$ attains its minimum for $n = 3k+1$.

(ii) Here is a generalization of (1). If all the coefficients of the polynomial P in (2) are positive; if S is equal to one of the three sets

$$\{n \mid n < p_1\}, \qquad \{n \mid n > p_k\}, \qquad \{n \mid n < p_1 \text{ or } n > p_k\};$$

and if $(n_1, n_2, \ldots, n_l) = \bar{n}(P)$, then the set

$$\{n_1, n_2, \ldots, n_l\} \cup \{n \mid p_1 \leqslant n \leqslant p_k\}$$

is a set of consecutive integers. (This statement follows from the results in [8] and the following observation: if the functions $\{u_j\}$, $j = 1, 2, \ldots, q$, form a Cartesian system on $[a, b]$ and if $v = \alpha_1 u_{k+1} + \cdots + \alpha_r u_{k+r}$ $(k \geqslant 0,\ r \geqslant 1,\ k + r \leqslant q)$, where $\alpha_s \geqslant 0$ for $s = 1, 2, \ldots, r$, then the system $u_1, \ldots, u_k, v, u_{k+r+1}, \ldots, u_q$ is also a Cartesian system on $[a, b]$.)

(iii) It is simplest to take for S the set of all nonnegative integers. In that case, $\pi_l(S)$ is the set of all polynomials of length $\leqslant l$, and the approximation of the polynomial P in $\pi_l(S)$ is trivial unless length $(P) > l$. However, if the last condition is satisfied, we are faced with the following problem: approximate P by polynomials of length $\leqslant l <$ length(P). This is a natural problem; it is also important. (If a good approximation to a function f by polynomials of length $\leqslant l$ were needed, a two step search could be attempted. Find a polynomial P which gives a good approximation fo f; then approximate P by polynomials of length $\leqslant l$.)

Accordingly, in the questions concerning approximation of a polynomial P by polynomials from $\pi_l(S)$, the special case $S =$ the set of all nonnegative integers, length$(P) > l$, appears to be the most important case.

3. The Main Question and Results

We have seen in 2(i) that when x^k is replaced by a polynomial, the straightforward generalization of (1) fails, and it fails badly. How badly can it fail in the worst case? Before stating that question more precisely it is useful to introduce abbreviations ba and uba for "best approximation" and "unique best approximation":

> If B is a normed vector-space, $f \in B$, $U \subset B$, then $\mathrm{ba}(f, U) = \{g \mid g \in U,\ \|f - g\| \leqslant \|f - h\| \text{ for every } h \in U\}$, and $\mathrm{uba}(f, U)$ is defined and equals g if and only if $g \in U$ and $\|f - g\| < \|f - h\|$ for every $h \in U$, $h \neq g$.

Question. Is there an $M_S(d, l) < \infty$ such that if $P \in \mathscr{P}_d$, $Q \in \mathrm{ba}(P, \pi_l(S))$, then $\deg Q \leqslant M_S(d, l)$?

The answer to this question depends on the norm. Namely, we can show that

(A) if $p < \infty$, the answer is yes, and

(B) if $p = \infty$ and $2l \leqslant d + 1$, the answer is no.

The proof of (A) will be published elsewhere; however, in the special case when $p = 2$, $[a, b] = [0, 1]$, and $l = 1$, that proof is particularly simple and has some geometrical flavor; therefore that special result is presented here.

THEOREM 1. *Given an infinite set S of nonnegative integers, there exists $M = M_S(d)$ such that if $P \in \mathscr{P}_d$ and $d_n = d_n(P) = \min_c \|P(x) + cx^n\|_{L_2(0,1)}$, $\mu = \min\{d_n | n \in S\}$, then $d_n > \mu$ if $n > M$.*

In the proof of Theorem 1 we shall use the following:

LEMMA. *If $K \subset R^m$, and if K is not contained in any proper subspace of R^m, then there exists $\mathscr{N} = \mathscr{N}(K) > 0$ such that*

$$\sup_{x \in K} |(x, y)| \geqslant \mathscr{N} |y| \qquad \textit{for every } y \in R^m.$$

(As one might expect, (,) above denotes the inner product in the Euclidean space R^m.)

Restated, the lemma becomes geometrically obvious. If a collection K of points in R^m does not lie in any hyperplane, then there exists $\delta > 0$ such that any strip between two parallel hyperplanes, which contains K, has width $\geqslant \delta$.

Proof of the Lemma. Without loss of generality we may assume that K is a finite set, so $\varphi(y) = \sup |(x, y)|$ is continuous. Set $\mathscr{N} = \min\{\varphi(y) \mid |y| = 1\}$.

We obtain (B) as a consequence of the following rather surprising fact.

THEOREM 2. *If $p = \infty$ and $2l \leqslant d + 1$, the mapping* uba: $\mathscr{P}_d \to \pi_l(S)$, *defined on a subset of $\mathscr{P}_d$, is a mapping onto $\pi_l(S)$.*

4. PROOFS

(i) *Proof of Theorem* 1. We may assume

$$\|P\|_{L_2(0,1)} = 1. \tag{4}$$

Then, by (3), $d_n^2 = 1 - (\sqrt{2n+1}\,(P, x^n))^2$, and so d_n takes its minimum value when $\lambda_n = \sqrt{2n+1}\,|(P, x^n)|$ takes its maximum value. Thus the theorem will be proved if we show

$$\max\{\lambda_n \mid n \in S\} > \max\{\lambda_n \mid n \in S, n > M\} \tag{5}$$

for some $M = M_s(d)$. We write

$$P(x) = \sum_{j=0}^{d} a_j x^j, \tag{6}$$

so that

$$\lambda_n = \sqrt{2n+1}\,\left|\sum_{j=0}^{d} \frac{a_j}{j+n+1}\right|. \tag{7}$$

Let $m = d+1$ and let a and v_n, $n = 0, 1,\ldots$, be points in R^m defined by $a = (a_0, a_1,\ldots, a_d)$, $v_n = (\sqrt{2n+1}/(n+1), \sqrt{2n+1}/(n+2),\ldots, \sqrt{2n+1}/(n+d+1))$. By (6) to every $P \in \mathscr{P}_d$ there corresponds a point $a \in R^m$. If the polynomial P satisfies (4), as we assume, the corresponding point a lies on the surface of a certain ellipsoid E in R^m. Thus there exist positive constants C_1 and C_2 such that

$$C_1 \leqslant |a| \leqslant C_2. \tag{8}$$

Using the well-known fact that a Cauchy determinant $1/(a_j + b_k)$, $1 \leqslant j, k \leqslant m$ is $\neq 0$ if all a_j's are distinct and all b_k's are distinct, it is easy to check that $\det(v_{n_1}, v_{n_2},\ldots, v_{n_m}) \neq 0$ if $n_1, n_2,\ldots, n_m$ are all distinct, i.e.,

if $n_1, n_2,\ldots, n_m$ are distinct, the vectors $v_{n_1}, v_{n_2},\ldots, v_{n_m}$ are linearly independent. (9)

From (7) we see that

$$\lambda_n = |(a, v_n)|. \tag{10}$$

The set $K = \{v_n \mid n \in S\}$, because of (9), does not lie in any proper subspace of R^m, so by the lemma, we obtain from (8) and (10) that

$$\sup\{\lambda_n \mid n \in S\} \geqslant \gamma, \tag{11}$$

where γ is a positive number dependent only on m and the set S. On the other hand, since $|(a, v_n)| \leqslant |a|\ |v_n| \leqslant C_2 |v_n| \to 0$, $n \to \infty$, we obtain that (a, v_n) tends to zero as $n \to \infty$, uniformly for $a \in E$. It follows that there exists $M = M_s(d)$ such that

$$\lambda_n = |(a, v_n)| \leqslant (\gamma/2) \qquad \text{if} \quad n > M. \tag{12}$$

The theorem is proved, since (11) and (12) imply (5).

(ii) *Proof of Theorem* 2. We shall prove this theorem by showing that if $Q \in \pi_l(S)$ and

$$\operatorname{uba}(Q, \mathscr{P}_d) = P, \tag{13}$$

then

$$\operatorname{uba}(P, \pi_l(S)) = Q. \tag{14}$$

By Chebyshev's theorem it follows from (13) that there exist x_ν, $\nu = 1, 2,..., d+2$ such that $a \leqslant x_1 < x_2 < \cdots < x_{d+2} \leqslant b$ and that

$$Q(x_\nu) - P(x_\nu) = \pm(-1)^\nu \|Q - P\|, \qquad \nu = 1, 2,..., d+2. \tag{15}$$

Assume that (14) does not hold. Then there exists $Q^* \in \pi_l(S)$, $Q^* \neq Q$ such that

$$\|Q^* - P\| \leqslant \|Q - P\|. \tag{16}$$

If $Q(x) = \sum_{j=0}^{l} a_j x^{n_j}$, $Q^*(x) = \sum_{j=0}^{l} b_j x^{m_j}$, consider the space V generated by $x^{n_1},..., x^{n_l}, x^{m_1},..., x^{m_l}$;

$$\text{both } Q \text{ and } Q^* \text{ belong to } V; \tag{17}$$

the dimension of V is $h \leqslant 2l$; the space V has a basis consisting of powers of x; and powers of x form a Chebyshev system on $[a, b]$ when $0 < a < b$. It follows that

$$\text{the best approximation of } P \text{ in } V \text{ is unique,} \tag{18}$$

and, since $d+1 \geqslant 2l \geqslant h$, we have $d+2 \geqslant h+1$, so that from (15) we get that

$$Q \text{ is a best approximation of } P \text{ in } V. \tag{19}$$

From (17)–(19) we obtain a contradiction with (16).

5. Two Open Problems

(i) A result concerning $\bar{n}(P)$, for polynomials P with positive coefficients, was given in 2(ii). Does that result hold if S is the set of all nonnegative integers and $l <$ length (P)? Does it hold if $S = \{n \mid n \neq p_j, j = 1, 2,..., k\}$?

(ii) The main question, raised in Section 3, is answered—for approximation in L_∞-norm—only in case $2l \leqslant d+1$. Even the following problem remains open:

If $\deg(P) = d \geqslant 2$ and Q is a best approximation to P in the set of all polynomials of length $\leqslant d$, is there an upper bound for $\deg(Q)$ which depends only on d?

REFERENCES

1. B. M. BAISHANSKI, Approximation by polynomials of given length, *Illinois J. Math.* **27** (3) (1983), 449–458.
2. B. M. BAISHANSKI AND R. BOJANIC, An estimate for coefficients of polynomials of given length, manuscript.
3. I. BOROSH, C. K. CHUI, AND P. W. SMITH, Best uniform approximation from a collection of subspaces, *Math. Z.* **156** (1977), 13–18.
4. I. BOROSH, C. K. CHUI, AND P. W. SMITH, On approximation of x^N by incomplete polynomials, *J. Approx. Theory* **24** (1978), 227–235.
5. G. G. LORENTZ, Approximation by incomplete polynomials (problems and results), *in* "Padé and Rational Approximation: Theory and Applications" (E. B. Saff and R. S. Varga, Eds.), Academic Press, New York, 1977.
6. E. B. SAFF AND R. S. VARGA, On lacunary incomplete polynomials, *Math. Z.* **177** (1981), 297–314.
7. E. B. SAFF AND R. S. VARGA, On incomplete polynomials, II, *Pacific J. Math.* **92** (1) (1981), 161–172.
8. O. SHISHA, Tchebysheff systems and best partial bases, *Pacific J. Math.* **86** (2) (1980) 579–592.
9. P. W. SMITH, An improvement theorem for Descartes systems, *Proc. Amer. Math. Soc.* **70** (1978), 26–30.